HEAT TRANSFER IN LAYERED SOILS BENEATH PARTIALLY INSULATED SLAB-ON-GRADE FLOORS

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ABSTRACT

An Analytical solution is developed to the problem of foundation heat transfer in non-homogeneous soils. In particular, the steady-state temperature field beneath partially insulated slab-on-grade floor is determined for multi-layered ground. A constant temperature water table is assumed to exist at a constant depth below the soil surface. The derived solution is applied to various slab insulation configurations to determine the effect of soil thermal characteristics on earth temperature distribution, heat flux along the slab, and total slab heat loss. The heat transfer between slabs and ground is found to vary as expected with the slab insulation configuration. When the ratio between the soil thermal conductivity of the top layer over that of the bottom layer is kept constant, it is found that the ground-coupling heat transfer is not strongly dependent on soil modeling characteristics.

INTRODUCTION

The thermal properties of the ground are generally recognized to be the most important parameters that affect foundation heat transfer. Unfortunately, data for soil thermal properties are often very difficult to obtain. Indeed, soil thermal properties are influenced by a myriad of factors such as soil type, soil density, soil moisture content, and even soil temperature [1-4]. These factors and consequently the soil thermal properties change with depth. However, very few methods exist that consider the spatial variation of soil thermal properties on foundation heat transfer.

The vast majority of the existing foundation heat transfer calculation methods assume that the soil is homogeneous with constant thermal properties [5-6]. The main reason of this simplification is to avoid the mathematical complexity associated with modeling heat transfer in non homogeneous medium. The few methods that consider some spatial variation of soil thermal conductivity are based on numerical techniques [7-8]. In particular, Mitalas method [8] considers an upper and lower layer of the ground medium with each layer characterized by a constant thermal conductivity. Typically, the upper layer has higher thermal conductivity than the lower layer because of the effect of the rainfall or frost.

In this paper, an analytical solution is derived for the problem heat transfer between non-homogeneous ground and a slab-on-grade floor with partial insulation configuration under steady-state conditions. The non-homogeneous ground is modeled as a series of soil layers with different thermal conductivities. The steady-state heat conduction equation is solved using the formalism of the ITPE technique [9-10]. The solution presented in this paper is the first analytical solution for slab-on-grade floors capable of analyzing the simultaneous effect on foundation heat transfer of (i) partial insulation configuration, (ii) water table depth, and (iii) non-isotropic earth.

This paper illustrates the effect of the spatial variation of soil thermal properties on soil temperature field, slab heat flux distribution, and total slab heat loss/gain for various commonly encountered slab insulation configuration: uninsulated slab, partially insulated slab, and uniformly insulated slab. In section 2 of the paper, the general solution of heat conduction equation in layered soil is developed. In section 3, the soil temperature field is illustrated for several slab insulation configurations and soil types. Section 4 deals with the heat flux distribution along the slab floor, and the total amount of heat flowing from the slab floor in contact with earth.

GENERAL SOLUTION

Figure 1 shows a model of a slab-on-grade floor above a multi-layered soil. A water table is assumed to exist at a given depth b with constant temperature T_w. The heat transfer between the ground and the air above the slab, kept at the temperature T_i, is modeled using a convective heat transfer coefficient *hi(x)*. This coefficient *hi(x)* represents the equivalent air-insulation-slab-soil conductance and may vary with the position along the slab floor. At the soil surface, a convective heat interchange between the soil and the ambient air is assumed to exist. The convection heat transfer coefficient h_a at the ground-air interface is function of several parameters such as the wind speed and the surface cover including vegetation and snow. The ambient air temperature T_a is readily available from the local weather data.

As illustrated in Fig. 1, the ground is modeled as a series of

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M layers with different thermal properties. Each layer (m), bounded by the surfaces $y = Y_m$ and $y = Y_{m+1}$ has a constant soil thermal conductivity k_m The domain of the problem is bounded by surfaces at $x = \pm L$ and $y = b$. The temperature at the surfaces $x = \pm L$ is assumed to be the undisturbed soil temperature, $T\infty(y)$, and varies as a piece-wise linear function of depth.

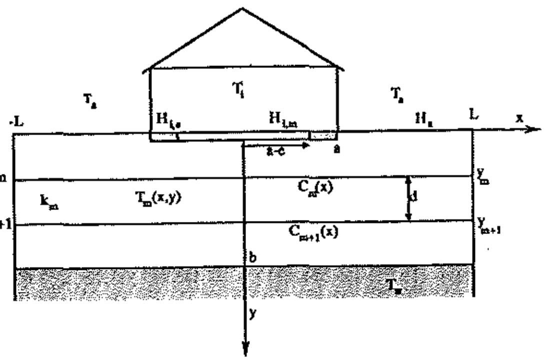


Figure 1 Model for the multilayered soil beneath a partially insulated slab-on-grade-floor.

The steady-state heat conduction equation in a non isotropic medium is given by the following equation:

$$\frac{\partial}{\partial x}(k_s\frac{\partial T(\mathbf{r},t)}{\partial x}) + \frac{\partial}{\partial y}(k_s\frac{\partial T(\mathbf{r},t)}{\partial y}) = 0 \quad (1)$$

The coordinates x and y are denoted by the vector space $\mathbf{r}$ (i.e. $\mathbf{r} = (x, y)$). The value of the soil thermal conductivity k_s is assumed constant in each zone [i.e., in zone(0), zone (I), etc. ...]. Therefore, the heat conduction equation (1) can be reduced in each zone to the Laplace equation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \quad (2)$$

The boundary conditions of equation (2) express (i) the continuity of heat flux at the surface $y = Y_{m+1}$ separating two adjacent layers, (ii) the convective heat flux at the soil surface (4) and (iii) the undisturbed soil temperature variation at $x = \pm L$:

$$-k_m \left.\frac{\partial T_m}{\partial y}\right|_{y=y_{m+1}} = -k_{m+1} \left.\frac{\partial T_{m+1}}{\partial y}\right|_{y=y_{m+1}}$$

$$k_o \left.\frac{\partial T_o}{\partial y}\right|_{y=o} = h(x)\,\{T_o - T_i(x)\}\,|_{y=o} \quad (3)$$

$$T(L,y) = T^{\infty}(y)$$

Using ITPE principles based on the separation of variables technique and the Fourier series expansion, the solution to Laplace's equation in zone (m) is:

$$T_m(x,y) = \frac{2}{L}\sum_{n=1}^{\infty} C_n^m\, cos\mu_n x\, \frac{sinh\mu_n(y_{m+1}-y)}{sinh\mu_n d}$$

$$+ \frac{2}{L}\sum_{n=1}^{\infty} C_n^{m+1}\, cos\mu_n x\, \frac{sinh\mu_n(y-y_m)}{sinh\mu_n d}$$

$$+ \frac{2}{d}\sum_{n=1}^{\infty} T_{n,m}^{\infty}\, sin\nu_n(y-y_m)\, \frac{cosh\nu_n x}{cosh\nu_n L}$$

Where

$$\nu_n = \frac{n\pi}{d} \qquad \mu_n = \frac{(2n-1)\pi}{2L}$$

$$T_{n,m}^{\infty} = \int_{y_m}^{y_{m+1}} T^{\infty}(y)\, sin\nu_n(y-y_m)\, dy$$

with $T\infty(y)$ defined as the undisturbed soil temperature profile. It can be shown that for any depth y such $Ym \leq Y \leq Y_{m+1}$ the undisturbed soil temperature is given as:

$$T^{\infty}(y) = T_w + \frac{\sum_{j=m}^{M-1} R_j}{R_{tot}}(T_a - T_w) + \frac{R_m}{R_{tot}}(T_w - T_a)(\frac{y}{d} - m)$$

with

$$R_j = \frac{d}{k_j} \qquad R_{tot} = \frac{1}{h_a} + \sum_{j=0}^{M-1} R_j$$

The first boundary condition that expresses the continuity of heat flux across the interface between each layer yields the following:

$$-k_m \left.\frac{\partial T_m}{\partial y}\right|_{y=y_{m+1}} = -k_{m+1} \left.\frac{\partial T_{m+1}}{\partial y}\right|_{y=y_{m+1}} \quad (5)$$

with

$$\frac{\partial T_m}{\partial y} = \frac{2}{L}\sum_{n=1}^{+\infty}(-\mu_n)C_n^m cos(\mu_n x)\,\frac{cosh\mu_n(y_{m+1}-y)}{sinh\mu_n d} + \frac{2}{d}\sum_{n=1}^{+\infty} T_{n,m}^{\infty}\nu_n cos\nu_n(y-y_m)\,\frac{cosh\nu_n x}{cosh\nu_n L}$$

$$+\frac{2}{L}\sum_{n=1}^{\infty}\mu_n C_n^{m+1}\, cos\mu_n x\,\frac{cosh\mu_n(y_{m+1}-y)}{sinh\mu_n d}$$

The coefficients C_n^m and C_n^{m+1} represent the Fourier coefficients of the temperature distribution at the interface of each layer. By multiplying both sides of eq. (3) by $cos(\mu_p x)$ and integrating from O to L, it can be shown that the coefficients C_p^{m+1} are subject to the following system of equations:

$$C_p^{m+1} = \alpha_p^{m+1} + \beta_p^{m+1}C_p^{m+1} + \gamma_p^{m+1}C_p^{m+2}$$

The coefficients $\alpha_p^{m+1}, \beta_p^{m+1}$ and γ_p^{m+1} are defined as follows:

$$\alpha_p^{m+1} = \frac{2}{d(k_m+k_{m+1})}\, coth\mu_p d \sum_{n=1}^{\infty}\frac{(-1)^p\nu_n}{\nu_n^2+\mu_p^2}\left\{k_m(-1)^n T_{n,m}^{\infty} - k_{m+1}T_{n,m}^{\infty}\right\}$$

$$\beta_p^{m+1} = \frac{k_m}{[k_m+k_{m+1}]cosh\mu_p d}$$

$$\gamma_p^{m+1} = \frac{k_{m+1}}{(k_m+k_{m+1})cosh\mu_p d}$$

These recursive relations (5) hold true for the coefficients of all boundaries, other than the top and bottom boundaries, which are denoted by C_p^M and C_p^0, where M is the total number of layers.

Since the temperature is fixed at T_w at the water table boundary, the coefficients C_p^M can be determined by the Fourier integral

$$C_p^M = \int_o^L T_w\, cos\mu_p x dx = -\frac{T_w(-1)^p}{\mu_p} \quad (6)$$

To determine the coefficients for $C_o(x)$, which is the temperature distribution at the soil surface $y=0$, the convective boundary condition is used. This boundary condition relates the conduction through the soil to the convection from the soil surface to the air above the ground surface:

$$
\begin{aligned}
k_o \frac{\partial T_o}{\partial y}\Big|_{y=o} &= h(x)\,\{T_o - T_{ia}(x)\}|_{y=o} \\
T_{ia}(x) &= \begin{cases} T_i & \text{if } |x| \le a \\ T_a & \text{if } |x| \ge a \end{cases} \\
h(x) &= \begin{cases} h_{i,m} & \text{if } |x| \le a-e \\ h_{i,e} & \text{if } a-e \le |x| \le a \\ h_a & \text{if } |x| \ge a \end{cases}
\end{aligned}
\tag{7}
$$

From the temperature expression (4), the temperature gradient $\frac{\partial T}{\partial y}$ can be determined and evaluated at $y = 0$. By setting this equal to the heat loss due to convection and multiplying both sides by $cos(\mu_p x)$ and integrating from 0 to L, a recursive relationship for C^0_p can be obtained:

$$
C_p^0 = \alpha_p^0 + \sum_{n=1}^{+\infty} \beta_{n,p}^0 C_n^0 + \gamma_p^0 C_p^1 \tag{8}
$$

with the coefficients α^0_p, $\beta^0_{n,}$, γ^0_p defined as follows:

$$
\alpha_p^0 = \frac{1}{(H_a^0 + \mu_p coth\mu_p d)} \left\{ \frac{-2}{d}\left(\sum_{n=1}^{\infty} T_{n,o}^{\infty} \frac{(-1)^p \nu_n \mu_p}{\nu_n^2 + \mu_p^2}\right) - \frac{(-1)^p H_a^0 T_a}{\mu_p} + \left(H_{i,e}^o T_i - H_a^0 T_a\right) F_p(a) + (H_{i,m}^0 T_i - H_{i,e}^0 T_i) F_p(e) \right\}
$$

where

$$
F_p(x) = \frac{1}{\mu_p} sin\mu_p x
$$

$$
\beta_{n,p}^0 = \frac{2}{L(H_a^0 + \mu_p coth\mu_p d)} \left\{ (H_a^0 - H_{i,e}^0) G_{n,p}(a) + (H_{i,e}^o - H_{i,m}) G_{n,p}(e) \right\}.
$$

where

$$
G_{n,p}(x) = \frac{1}{2}\left(\frac{sin(\mu_n - \mu_p)x}{\mu_n - \mu_p} - \frac{sin(\mu_n + \mu_p)}{(\mu_n + \mu_p)}\right)
$$

$$
\gamma_p^0 = \frac{\mu_p}{sinh\mu_p d(H_a^0 + \mu_p coth\mu_p d)}
$$

where

$$
H_a^0 = \frac{h_a}{k_0}; \quad H_{i,m}^0 = \frac{h_{i,m}}{k_0}; \quad H_{i,e}^0 = \frac{h_{i,e}}{k_0}
$$

The coefficients C^n_p are determined by (i) truncating the sum in equation (8) to N terms, and (ii) solving the system of equations (5), (6), and (8) using the Gauss-Jordan elimination method. In section 4, the effect of the number N on the accuracy of the results will be discussed in detail.

SOIL TEMPERATURE DISTRIBUTION

Figures 2 through 10 show the soil temperature field beneath a slab-on-grade floor. The slab has a width of $2a$ = 10 m (32 ft). The interior air temperature above the slab is kept constant at T_i = 20 °C (68 °F), while the ambient air temperature is assumed to be T_a = 15 °C (59 °F). The convective heat transfer coefficient at the soil surface is set at h_a = 5 W/m^2 °C (0.88 $Btu/hr\,ft^2$ °F). A water table exists at a depth of b = 5 m (16.4 ft) and has a constant temperature T_w = 10 °C (50 °F). Three insulation configurations and three soil conditions are considered. The insulation configurations are:

(i) Uninsulated slab [$h_{i,m} = h_{i,e}$ = 4 W/m^2 °C (0.70 $Btu/hr\,ft^2$ °F)]
(ii) Partially insulated slab [$h_{i,m}$ = 4 W/m^2 (0.70 $Btu/hr\,ft^2$ °F); $h_{i,e}$ = 0.5 W/m^2 °C (0.09 $Btu/hr\,ft^2$ °F)]
(iii) Uniformly insulated slab [$h_{i,m} = h_{i,e}$ = 1 W/m^2 °C (0.18 $Btu/hr\,ft^2$ °F)]

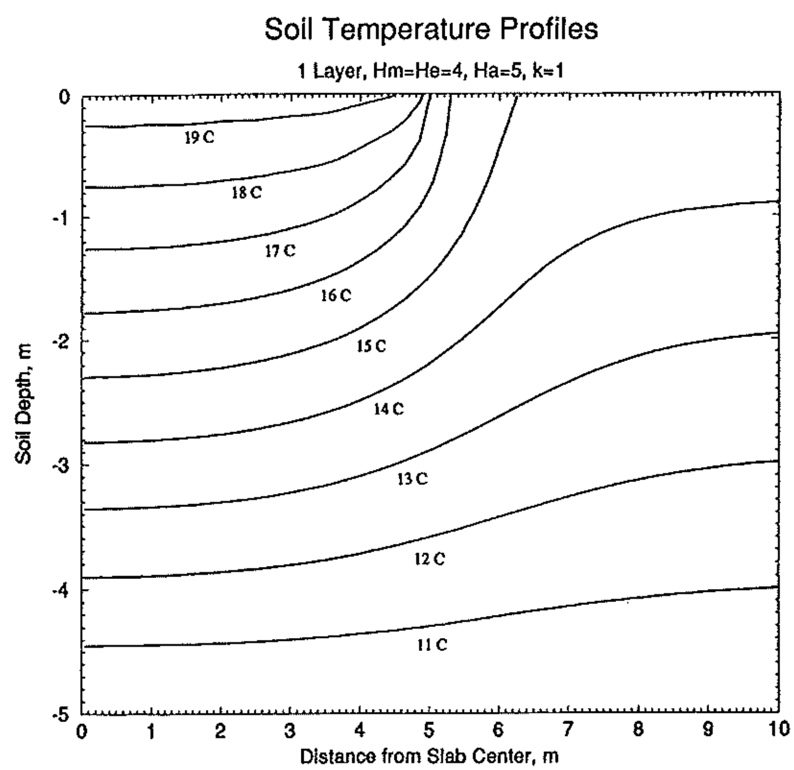
The three soil configurations are as follows:

(a) Homogeneous soil [k_0 = 1.0 W/m °C (0.58 $Btu/hr\,ft$ °F)].
(b) Two-layered soil [the top layer located between y = 0 m (0 ft) and y = 1 m (3.28 ft) the ground thermal conductivity is k_0 = 2.0 W/m °C (1.16 $Btu/hr\,ft$ °F) and the bottom layer between y = 1 m (3.28 ft) and y = 5 m (16.4 ft) has a thermal conductivity of k_1 = 1.0 W/m °C (0.58 $Btu/hr\,ft$ °F)]. This condition may occur after rainfall.
(c) Five-layered soil [with k_o = 3.0 W/m °C (1.73 $Btu/hr\,ft$ °F) and $y1$ = 1 m (3.28 ft); k_1 = 2.5 W/m °C (1.44 $Btu/hr\,ft$ °F) and $y2$ = 2 m (6.56 ft); k_2 = 2.0 W/m °C (1.16 $Btu/hr\,ft$ °F) and $y3$ = 3 m (9.84 ft); k_3 = 1.5 W/m °C (0.87 $Btu/hr\,ft$ °F) and $y4$ = 4 m (13.12 ft); k_4 = 1.0 W/m °C (0.58 $Btu/hr\,ft$ °F) and $y5$ = 5 m (16 4 ft)].

Using the solution developed in section 2, the temperature isotherms within the ground were generated. To insure accurate solution, the number of iterations N was selected to be N = 200. Increasing the value of N has no significant effect on the temperature calculation.

Figures 2-10 provide valuable insight on the effect of both soil conditions and insulation configurations on soil temperature distribution. Generally, the insulation reduces the magni-

Figure 2 Temperature isotherms for homogeneous soil beneath uninsulated slab.



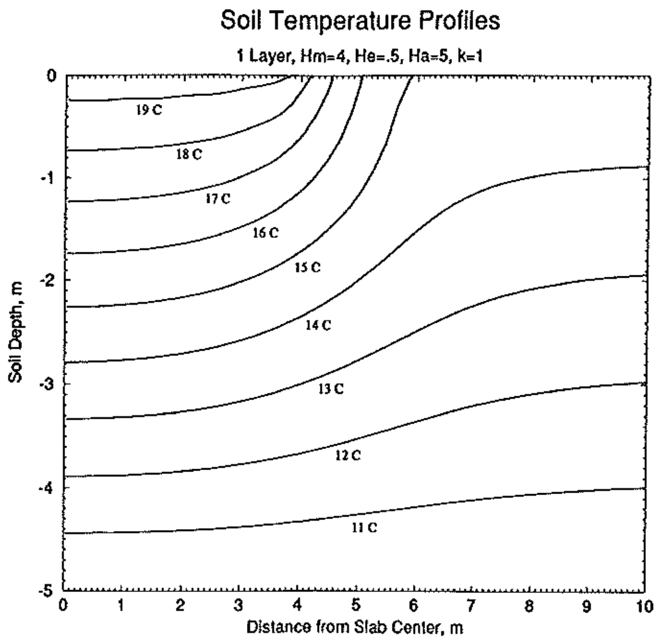


***Figure 3** Temperature isotherms for two-layered soil beneath uninsulated slab.*

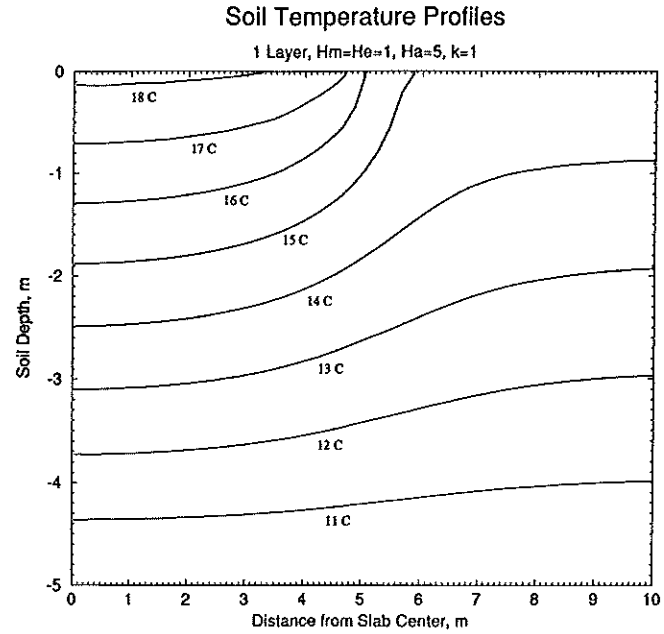


***Figure 4** Temperature isotherms for five-layered soil beneath uninsulated slab.*

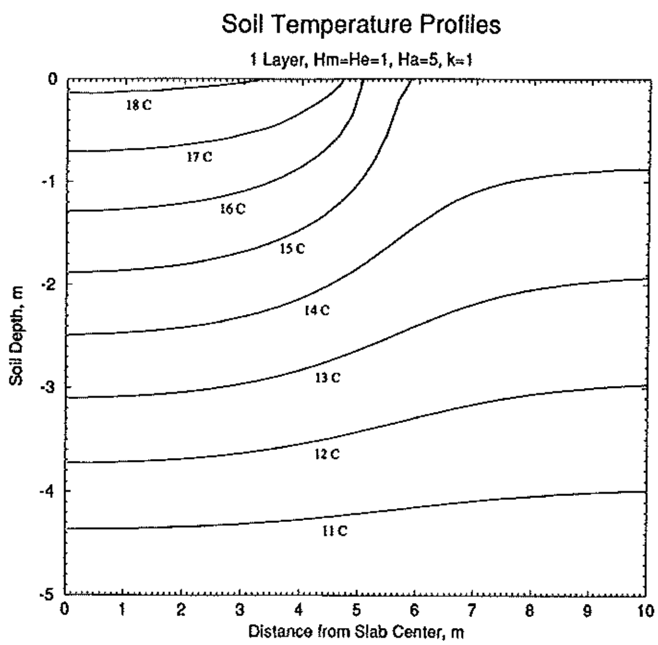


***Figure 4** Temperature isotherms for five-layered soil beneath uninsulated slab.*

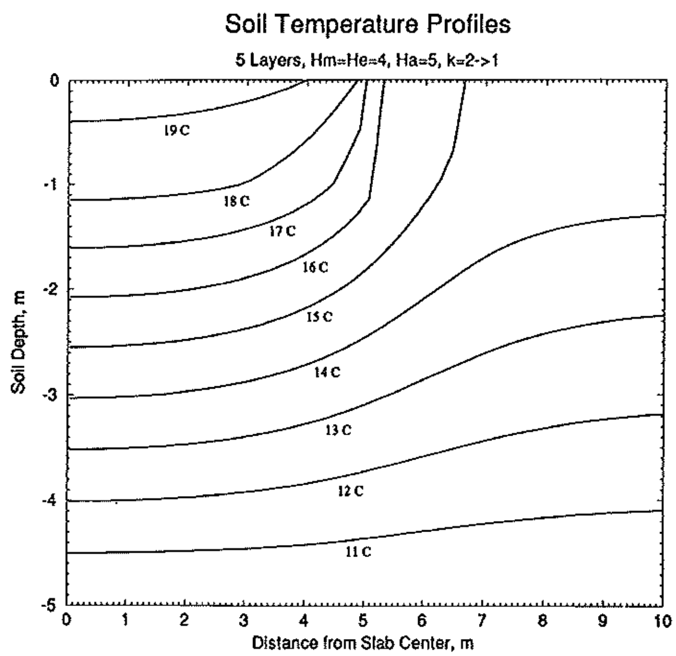


***Figure 5** Temperature isotherms for homogeneous soil beneath partially insulated slab.*

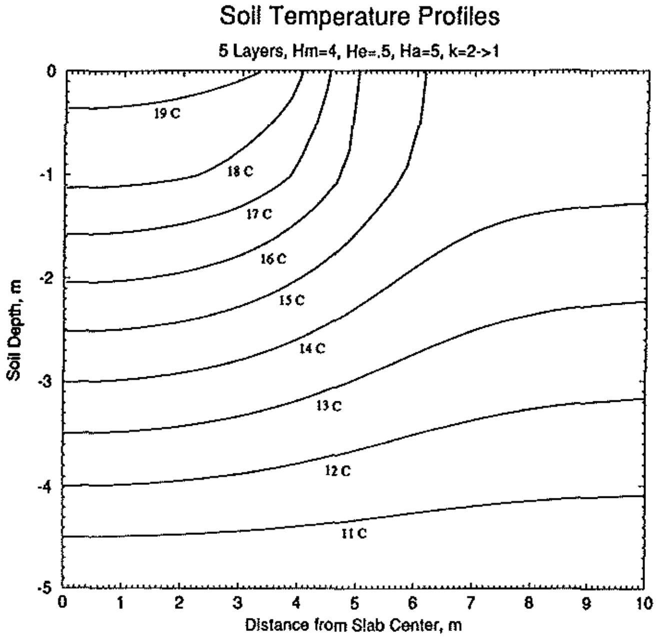


Figure 6 Temperature isotherms for two-layered soil beneath partially insulated slab.

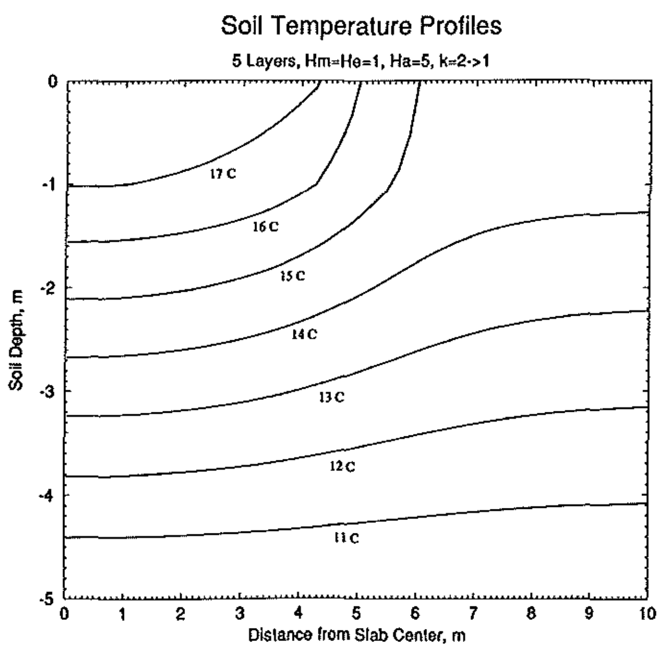


Figure 7 Temperature isotherms for five-layered soil beneath partially insulated slab.

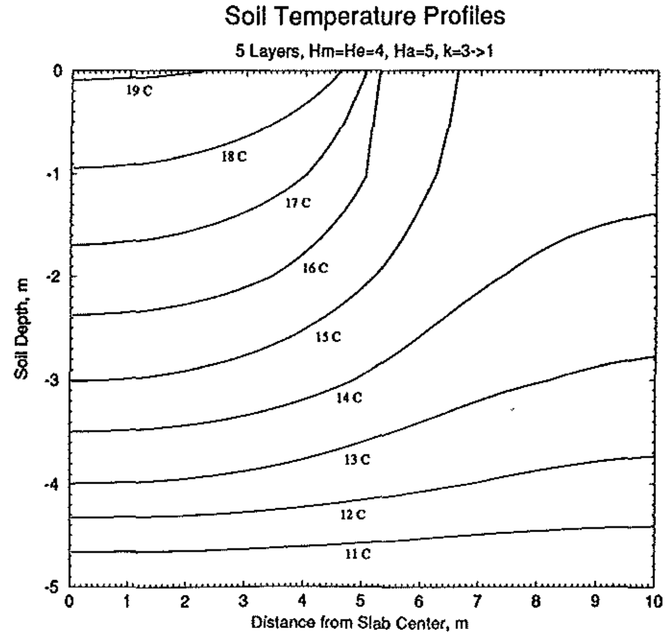


Figure 8 Temperature isotherms for homogeneous soil beneath uniformly insulated slab.

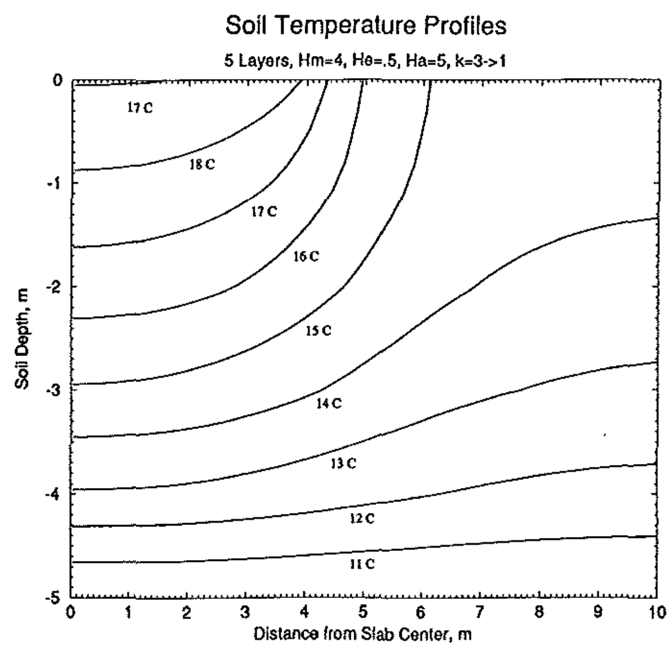


Figure 9 Temperature isotherms for two-layered soil beneath uniformly insulated slab.

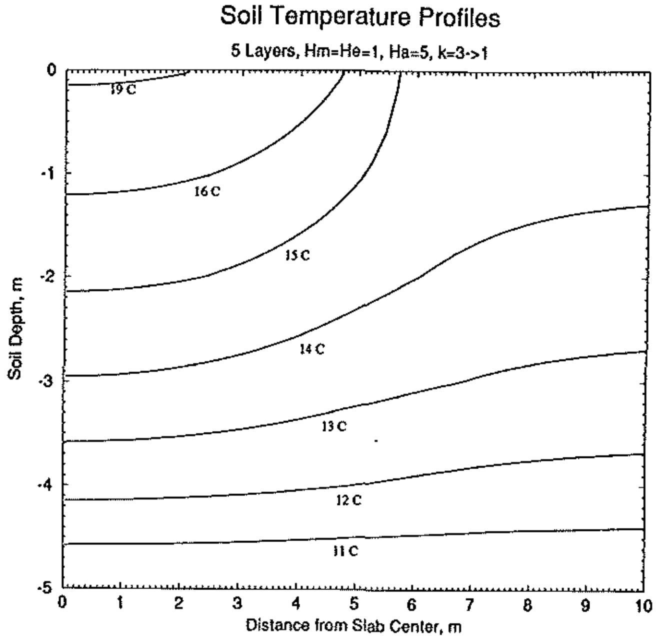


Figure 10 Temperature isotherms for five-layered soil beneath uniformly insulated slab.

tude of temperature change along the slab surface. In case of isotropic soil and for the uninsulated slab, the slab surface temperature varies abruptly from 19 °*C* (66 °*F*) to about 15 °*C* (59 °*F*) within less than 0.5 *m* (1.64 *ft*) at the slab edge. Meanwhile for the same isotropic soil and for uniformly insulated slab, the slab surface temperature drops only from 18 °*C* (64 °*F*) to 15 °*C* (59 °*F*) along 2 *m* (6.56 *ft*) wide area of the slab edge. Thus, the insulation affects the slab surface in two ways: (i) decreases the overall slab surface temperature and (ii) smoothes the temperature variation along the slab surface.

The effect of the soil conditions is evident when comparing the soil temperature profiles in the uninsulated slab case in Figs. 5 through 7. As the soil thermal conductivity of the top layer increases, the soil temperature changes with the characteristics of the ground. Indeed, in the five-layered ground (see Fig. 7), the earth temperature just beneath the slab center is generally lower than that in the homogeneous ground (see Fig. 5). However, in the case of two-layered soil (see Fig. 6), the opposite trend is observed, that is the soil temperature below the slab is warmer than that of the isotropic soil. The thermal conductivity of the top layer in the two-layered soil is double that of the homogeneous soil. Therefore, more than its magnitude. the relative change of soil thermal conductivity from one layer to the other has significant effect on the temperature field within the ground.

HEAT TRANSFER FROM SLAB

Slab Heat Flux Distribution

The heat flux variation, $I(x)$, along the slab surface is determined using the following equation:

$$I(x) = h(x)(T(x,0.01) - T_i) \tag{9}$$

The value of I(x) would normally be evaluated at the slab boundary (i.e., $y = 0$) where the convection heat transfer occurs. However, to avoid any convergence problems, the heat flux is instead evaluated at a depth that is infinitesimally close to the slab surface [$y = 0.01\ m$ .03 *ft*)].

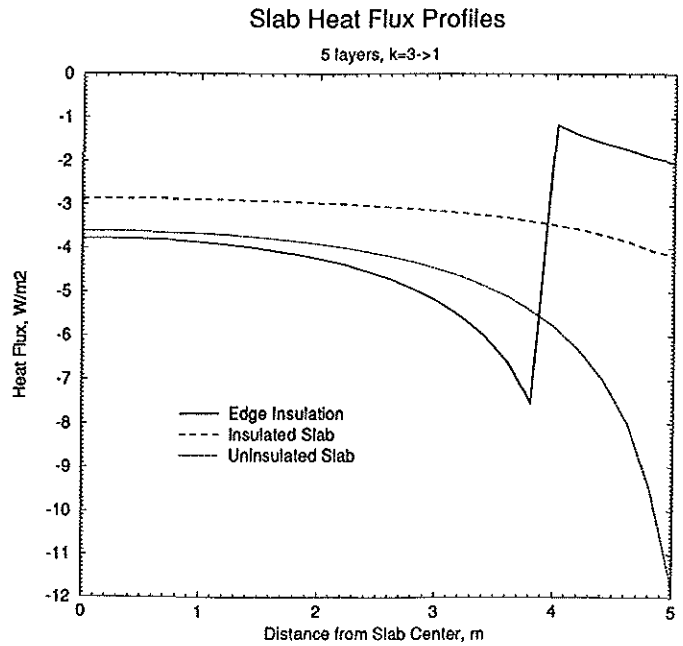


Figure 11 Heat flux profiles along the slab surface for five-layered soil.

The heat flux as a function of the distance along the slab is presented in Figure 11 for the five-layered soil. The slab width is $2a = 10\ m$ (32.8 *ft*), the air temperature above the slab is T_i = 20 °*C* (68 °*F*), the ambient temperature is T_a = 15 °*C* (59 °*F*), and the water table temperature is T_w = 10 °*C* (50 °*F*). The three insulation configurations listed in section 3 are considered in Fig. 11. As expected, the heat flux is generally the highest in the uninsulated part of the slab. It is interesting to note that when adding perimeter insulation, heat losses are reduced at the slab edge. However, an increase of heat flux occurs at the center of the slab especially around the edge of the perimeter insulation. In all the insulation configurations, corner effects are evident by the high heat flux at the slab edges. In the central area of the slab, the heat flux is almost constant.

Total Slab Heat Transfer

The total slab heat loss, Q, is obtained by integrating the heat flux $I(x)$, evaluated at $y = 0$, over the interval [-a, a]:

$$Q = \frac{4}{L}\sum_{n=1}^{\infty}\frac{C_n^0}{\mu_n}\{(h_{i,m} - h_{i,e})sin\mu_n(a-e) + h_{i,e}sin\mu_n a\} - 2T_i\{h_{i,m}(a-e) - h_{i,e}e\} \tag{10}$$

Figure 12 show the effect of the number of iterations N on the calculation of the total heat loss, Q, for partially insulated slab. Three soil layer configurations are considered in Fig. 1. These configurations are described in section 3 and represent isotropic, two-layered, and five-layered soils. The same slab characteristics of Fig. 11 are used in Fig. 12. For each soil configuration, the total slab heat loss approaches an asymptotic value as the number N increases. Although the asymptotic value is different for each soil configuration, the asymptotic nature of the graphs is similar: the total slab heat loss reaches a constant value when the number of iterations N is above 30. A further increase in N provides no significant increase in the accuracy of the slab heat loss calculation.

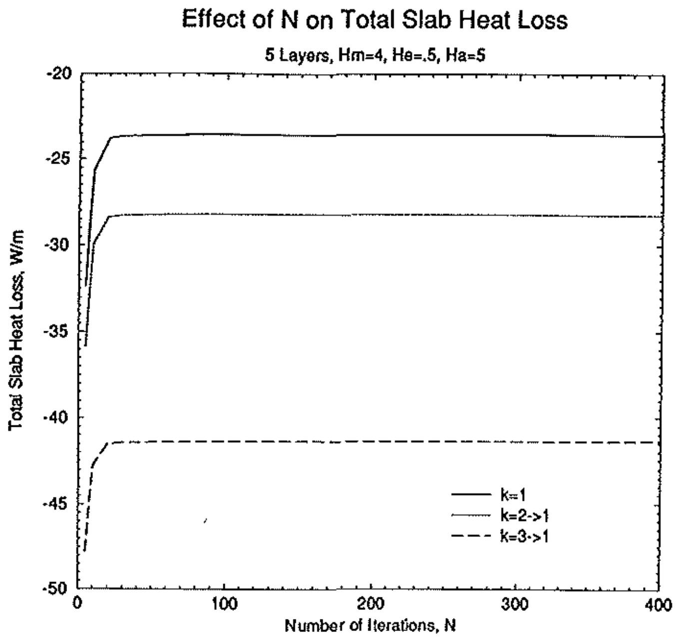


Figure 12 Effect of the number of iterations N on the total slab heat loss calculation from partially insulated slab.

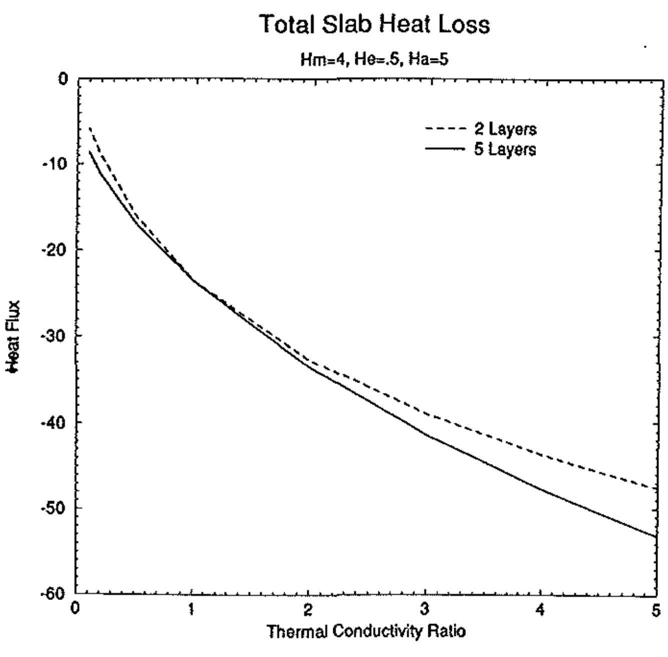


Figure 13 Effect of soil thermal conductivity ratio on the total slab heat loss from partially insulated slab.

Figure 13 provides an analysis of the effect of layered soil on the total slab heat loss. The slab of width $2a = 10\ m$ ($32.8\ ft$) is assumed to be partially insulated. Soils with two and five layers are considered in Fig. 13. The total slab heat loss variation with the thermal conductivity ratio between top layer and bottom layer is shown for two-layered and five-layered soils. The soil thermal conductivity of the bottom layer is held constant and is set equal to 1.0 $W/m\ °C$ (0.58 $Btu/hr\ ft\ °F$), while the thermal conductivity of the layers above the bottom layer is allowed to vary to generate the desired thermal conductivity ratio. For the soil with five layers, the thermal conductivity is defined to vary linearly from the top layer to the bottom layer for each thermal conductivity ratio. For the two-layered soil, the thermal conductivity is set equal to 1.0 $W/m\ °C$ (0.58 $Btu/hr\ ft\ °F$) for the bottom half of the soil depth, while the thermal conductivity of the top half of the soil depth was allowed to vary to give the required thermal conductivity ratio.

As illustrated in Figure 13, for all the thermal conductivity ratios, the five-layered soil allows a larger total slab heat loss than the two-layered soil. This result implies that for a given thermal conductivity ratio, the five-layered soil has lower average thermal resistance than the two-layered soil. In the particular case of thermal conductivity ratio of 1, the two soils are isotropic and have the same slab heat loss. Therefore, for all values of thermal conductivity ratio, a linear change of conductivity through five layers produces a lower overall thermal resistance and greater heat transfer than a slab consisting of two layers and a step change in soil thermal conductivity. In general, it can be shown that the thermal resistance of soil modeled with four or more layers has always a thermal resistance less than or equal to soil modeled with two layers for a given thermal conductivity ratio. The response of the slab is similar for both multi-layered soil models when compared to the isotropic soil model. For instance, when the thermal conductivity ratio is equal to 0.5, the total slab heat loss is higher by 39 % for the two-layered soil and by 33 % for the five-layered soil compared to the heat loss experienced by the slab in an isotropic soil. However, when the thermal conductivity ratio is 2, the losses from the slab are lower by 35 % for the two-layered soil, and by 39 % for the five-layered soil. It is interesting to note that although the difference in slab heat loss of the two non-homogeneous soils appears larger for the thermal conductivity ratios greater than 1, the percentage difference between the two types of soil is actually greater for thermal conductivity ratios less than 1. This difference is only 3 % for soil thermal conductivity ratio of 2, but it is more than 10 % for soil thermal conductivity ratio of 0.5.

SUMMARY AND CONCLUSIONS

An analytical solution has been developed to determine the slab-on-grade floor heat losses in multi-layered soil for various insulation configurations. The results show that the temperature distribution is strongly affected by how the slab is insulated and how the soil is modeled. The thermal resistance of soil modeled as a five-layered medium was found to